Follow-up on M/L/N Languages

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On Admissibility and Derivability

Recapping from Tuesday's lecture:

Given a unary judgement $\mathbf{J}$ defined by inference rules $I_1 \ldots I_n$, a new inference rule I_x is

- *derivable* if the conclusion of I_x can be derived via rules $I_1 \ldots I_n$ from its premises as local axioms
- *admissible* if the judgement $\mathbf{J}'$ defined by $I_1 \ldots I_n, I_x$ is true for the same set of values as $\mathbf{J}$

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Note the meta-logic/logic distinction. We prove admissibility in the meta-logic of these slides.

Recall M, L, N

Recall our language of matched parentheses

$$\{\epsilon, (), (()), ()(), \ldots\}$$

Ambiguous Definition

$$\boxed{s \ \mathbf{M}}$$

$$\frac{}{\varepsilon \ \mathbf{M}} M_E \qquad \frac{s \ \mathbf{M}}{(s) \ \mathbf{M}} M_N \qquad \frac{s_1 \ \mathbf{M} \qquad s_2 \ \mathbf{M}}{s_1 s_2 \ \mathbf{M}} M_J$$

Unambiguous Definition

$$\boxed{s\ \mathbf{L}} \qquad \boxed{s\ \mathbf{N}}$$

$$\frac{}{\varepsilon\ \mathbf{L}}L_E \qquad \frac{s\ \mathbf{L}}{(s)\ \mathbf{N}}N_N \qquad \frac{s_1\ \mathbf{N} \qquad s_2\ \mathbf{L}}{s_1 s_2\ \mathbf{L}}L_J$$

On Admissibility into **L**

In Tuesday's lecture we tried to prove that **M** and **L** have the same contents. Proving **L** is contained in **M** requires a simultaneous induction on the **L**/**N** definition, but each of the cases is easy. The tutorials went over this again.

We then looked at the proof that **M** is contained in **L**. Some of the tutors attempted this as well.

Note that this direction is equivalent to showing that each of the rules of **M** would be admissible to **L**. M_E is already present, and M_N is derivable, but M_J is not derivable. We need to do more logical work to show it.

On Induction and Meta-Logic

In the notes I typed out last lecture there was some confusion about induction and the meta-logic.

Gentzen's Natural Deduction calculus gives us a meta-logic we are using to formalise the syntax and semantics of programs. The object logic is the inference rules.

The rule induction is in the meta-logic, which is more general. We can do induction to prove properties that can't be stated as inference rules.

For instance, we could prove two languages/judgements do not overlap by proving that whenever $s\ \mathbf{J}_1$ holds, $\neg(s\ \mathbf{J}_2)$. Inference rules cannot be phrased in terms of negation, but an inductive proof about them can.

Lemma 1

$$\frac{s \ \mathbf{L} \qquad t \ \mathbf{L}}{st \ \mathbf{L}}$$

We prove this lemma by rule induction on the derivation that s is in $\mathbf{L}$, and for all t. That is, we prove

$$\forall\ s.\ s\ \mathbf{L} \longrightarrow (\forall\ t.\ t\ \mathbf{L} \longrightarrow st\ \mathbf{L})$$

by induction on $s\ \mathbf{L}$ with $P(s) \equiv (\forall\ t.\ t\ \mathbf{L} \longrightarrow st\ \mathbf{L})$.

Prove $\forall\ s.\ s\ \mathbf{L}\ \longrightarrow (\forall\ t.\ t\ \mathbf{L}\ \longrightarrow\ st\ \mathbf{L})$
by induction on $s\ \mathbf{L}$ with $P_0(s) \equiv (\forall\ t.\ t\ \mathbf{L}\ \longrightarrow\ st\ \mathbf{L})$.

More precisely, we use the $\mathbf{L}/\mathbf{N}$ simultaneous induction with $P(s) \equiv P_0(s) \wedge s\ \mathbf{L}$, $Q(s) \equiv s\ \mathbf{N}$.

Only the P_0 conclusions are tricky, the $s\ \mathbf{L}$ and $s\ \mathbf{N}$ goals are easy.

In the base case $P_0(\epsilon)$: $(\forall\ t.\ t\ \mathbf{L}\ \longrightarrow\ \epsilon t\ \mathbf{L})$.

Inductive case, from L_J:

Assume IHs $s_1\ \mathbf{N}$, $s_2\ \mathbf{L}$ and $P_0(s_2)$.

Show $P(s_1 s_2) \equiv (\forall\ t.\ t\ \mathbf{L}\ \longrightarrow\ s_1 s_2 t\ \mathbf{L}$.

The inductive case for N_N is just $s\ \mathbf{N}$ and is easy.